TRIGONOMETRY

01. BASIC TRIGONOMETRY

1. $\sin^2\theta + \cos^2\theta = 1$

2. $1 + \tan^2\theta = \sec^2\theta$

3. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

4. $\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$

5. $\operatorname{cosec}\theta + \cot\theta = \frac{1}{\operatorname{cosec}\theta - \cot\theta}$

6. $\sin(-\theta) = -\sin\theta, \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

   $\tan(-\theta) = -\tan\theta, \cot(-\theta) = -\cot\theta$

   $\cos(-\theta) = \cos\theta, \sec(-\theta) = \sec\theta$

7. $\sin(A+B) = \sin A\cos B + \cos A\sin B$

   $\sin(A-B) = \sin A\cos B - \cos A\sin B$

   $\cos(A+B) = \cos A\cos B - \sin A\sin B$

   $\cos(A-B) = \cos A\cos B + \sin A\sin B$

8. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

   $\sin(A+B)\sin(A-B) = \cos^2 B - \cos^2 A$

9. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

   $\cos(A+B)\cos(A-B) = \cos^2 B - \sin^2 A$

10. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

11. $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

12. $\tan\left(\frac{\pi}{4} + A\right) = \frac{1+\tan A}{1-\tan A} = \frac{\cos A + \sin A}{\cos A - \sin A}$

13. $\tan\left(\frac{\pi}{4} - A\right) = \frac{1-\tan A}{1+\tan A} = \frac{\cos A - \sin A}{\cos A + \sin A}$

14 $\cot(A+B) = \frac{\cot B\cot A - 1}{\cot B + \cot A}$

15. $\cot(A-B) = \frac{\cot B\cot A + 1}{\cot B - \cot A}$

16. $\sin 2A = 2\sin A\cos A$

    $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

17. $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

18. $\cot 2A = \frac{\cot^2 A - 1}{2\cot A}$

19. $\cos 2A = \cos^2 A - \sin^2 A$

    $\cos 2A = 1 - 2\sin^2 A$, $\cos 2A = 2\cos^2 A - 1$

    $2\sin^2 A = 1 - \cos 2A$, $2\cos^2 A = 1 + \cos 2A$

    $\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}$, $\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}$

    $\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

20.
$\sin(A+B+C) = \sum\sin A\cos B\cos C - \sin A\sin B\sin C$

21.
$\cos(A+B+C) = \cos A\cos B\cos C - \sum\cos A\sin B\sin C$

22. $\tan(A+B+C) = \frac{\sum\tan A - \prod\tan A}{1 - \sum\tan A\tan B} = \frac{S_1 - S_3}{1 - S_2}$

    $\cot(A+B+C) = \frac{\sum\cot A - \prod\cot A}{1 - \sum\cot A\cot B} = \frac{S_1 - S_3}{1 - S_2}$

23. $\sin 3A = 3\sin A - 4\sin^3 A$

24. $\cos 3A = 4\cos^3 A - 3\cos A$

25. $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

    $\cot 3A = \frac{3\cot A - \cot^3 A}{1 - 3\cot^2 A}$

26. $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

27. $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

28. $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

29. $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

30. $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

31. $\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$

32. $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

33. $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

34. $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

35. $\tan A + \tan B = \frac{\sin(A+B)}{\cos A\cos B}$

36. $\tan A - \tan B = \dfrac{\sin(A-B)}{\cos A \cos B}$

37. $\cot A + \tan A = 2\operatorname{cosec} 2A$,
$\cot A - \tan A = 2\cot 2A$

38. $\sin 15^o = \cos 75^o = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$\sin 75^o = \cos 15^o = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

$\tan 15^o = \cot 75^o = 2-\sqrt{3}$

$\tan 75^o = \cot 15^o = 2+\sqrt{3}$

$\tan 22\frac{1}{2}^o = \cot 67\frac{1}{2}^o = \sqrt{2}-1$

$\tan 67\frac{1}{2}^o = \cot 22\frac{1}{2}^o = \sqrt{2}+1$

$\sin 18^o = \cos 72^o = \dfrac{\sqrt{5}-1}{4}$

$\sin 36^o = \cos 54^o = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

$\sin 54^o = \cos 36^o = \dfrac{\sqrt{5}+1}{4}$

$\sin 72^o = \cos 18^o = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

39. $\sin A \sin(60-A)\sin(60+A) = \dfrac{1}{4}\sin 3A$

$\cos A \cos(60-A)\cos(60+A) = \dfrac{1}{4}\cos 3A$

$\tan A \tan(60-A)\tan(60+A) = \tan 3A$

40. $-1 \le \sin A \le 1$ and $-1 \le \cos A \le 1$
$-\infty < \tan A < \infty$ and $-\infty < \cot A < \infty$

$\operatorname{cosec} A \le -1,\ \operatorname{cosec} A \ge 1$
$\sec A \le -1,\ \sec A \ge 1$

41. If $A+B+C=\pi$ then

$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

$\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

$\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1$

$\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

42. For the expression $a\cos x + b\sin x + c$

Max. value = $c + \sqrt{a^2+b^2}$

Min. value = $c - \sqrt{a^2+b^2}$

43. Period of $\sin x, \cos x, \operatorname{cosec} x, \sec x$ is 2π

Period of $\tan x, \cot x$ is π

Period of $\sin^m x$ or $\cos^m x = \begin{cases} \pi & \text{m is even} \\ 2\pi & \text{m is odd} \end{cases}$

44. Period of $f(x)$ is P,

Then period of $f(ax+b)$ is $\dfrac{P}{|a|}$.

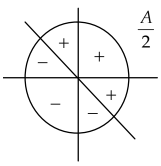
45. Period of $af(x) \pm bg(x)$ is less than or equal to L.C.M of {period of $f(x)$, period of $g(x)$}

46. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... + ... + \sin(\alpha+(n-1)\beta) = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\left[\sin\left(\dfrac{\alpha+\alpha+(n-1)\beta}{2}\right)\right]$

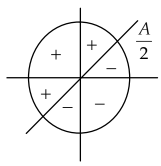
47. $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ... + ... + \cos(\alpha+(n-1)\beta) = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\left[\cos\left(\dfrac{\alpha+\alpha+(n-1)\beta}{2}\right)\right]$

48. $\cos A.\cos 2A.\cos 4A.\cos 8A......\cos 2^n A = \dfrac{\sin 2^{n+1}A}{2^{n+1}\sin A}$

49. $\sin\dfrac{A}{2} + \cos\dfrac{A}{2} = \pm\sqrt{1+\sin A}$,



50. $\sin\dfrac{A}{2} - \cos\dfrac{A}{2} = \pm\sqrt{1-\sin A}$



02. TRIGONOMETRY EQUATIONS

Syllabus : *Principle values and General solutions of Trigonometric equations in various forms.*

GENERAL SOLUTIONS

1 $\sin\theta = 0 \Leftrightarrow \theta = n\pi$

2 $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}$

3 $\tan\theta = 0 \Leftrightarrow \theta = n\pi$

4 $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

5 $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$

6 $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

7 $\sin^2\theta = \sin^2\alpha, \cos^2\theta = \cos^2\alpha, \tan^2\theta = \tan^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$

8 $\sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}$

9 $\cos\theta = 1 \Leftrightarrow \theta = 2n\pi$

10 $\cos\theta = -1 \Leftrightarrow \theta = (2n+1)\pi$

11 $\sin\theta = \sin\alpha$ and $\cos\theta$ = $\cos\alpha \Leftrightarrow \theta = 2n\pi + \alpha$

Note :

1 Every where in this chapter '*n*' is taken as an integer, if not stated otherwise.

2 The general solution should be given unless the solution is required in a specified interval or range.

3 α is taken as the principal value of the angle. (i.e. Numerically least angle is called the principal value).

03. INVERSE TRIGONOMETRY FUNCTIONS

Syllabus : *Domain and range of inverse trigonometric functions, Sum and Difference formulae and solutions of inverse trigonometric equations.*

INVERSE CIRCULAR FUNCTIONS

1. $\sin^{-1} x + \cos^{-1} x = \pi/2$, $\tan^{-1} x + \cot^{-1} x = \pi/2$, $\sec^{-1} x + \text{cosec}^{-1} x = \pi/2$.

2. If $\sin^{-1} x + \sin^{-1} y = \pi/2$, then $x^2 + y^2 = 1$.

3. $\sin(\cos^{-1} x) = \sqrt{1-x^2}$, $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

4. If $-1 < x < 1$, $\sin^{-1} \frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1} x$

5. If $x^2 + y^2 \le 1$, $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$

If $x^2 + y^2 > 1$, $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$.

$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$

6. (a) If $-1 < x < 1$, $-1 < y < 1$ and $xy < 1$ then

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$$

(b) If $x > 0$, $y > 0$ and $xy > 1$ then

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy} + \pi$$

(c) If $x < 0$, $y < 0$ and $xy > 1$, then

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy} - \pi$$

7. (a) If $xy > -1$ then $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$

(b) If $x > 0$, $y < 0$ and $xy < -1$, then $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy} + \pi$

(c) If $x < 0$, $y>0$ and $xy < -1$ then, $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy} - \pi$

04. PROPERTIES OF TRIANGLE

Syllabus : ***Relation between sides and angles of a triangle, since rule, cosine rule, projection rule, Napier's rule, half-angle formula and the area of a triangle.***

PROPERTIES OF TRIANGLE

1. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ i.e. the sides of a triangle are proportional to the sines of the opposite angles.

2. $a = b\cos C + c\cos B$
$b = c\cos A + a\cos C$
$c = a\cos B + b\cos A$

3. $a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = c^2 + a^2 - 2ca\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

4. $2bc\cos A = b^2 + c^2 - a^2$, $2ca\cos B = c^2 + a^2 - b^2$, $2ab\cos C = a^2 + b^2 - c^2$

5. $\cos A = \frac{b^2+c^2-a^2}{2bc},\ \cos B = \frac{c^2+a^2-b^2}{2ca},\ \cos C = \frac{a^2+b^2-c^2}{2ab}$

6. $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(using symmetry, it is better to form the same for other angles)

7. $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$

$= \sqrt{s(s-a)(s-b)(s-c)}$

$= 2R^2 \sin A \sin B \sin C$

$= \frac{abc}{4R}$

$= \frac{1}{2}\frac{a^2\sin B\sin C}{\sin(B+C)} = \frac{1}{2}\frac{b^2\sin C\sin A}{\sin(C+A)} = \frac{1}{2}\frac{c^2\sin A\sin B}{\sin(A+B)}$

8. $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)},\ \tan\frac{B}{2} = \frac{\Delta}{s(s-b)},\ \tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$

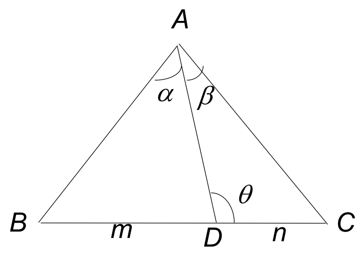
9. $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = (s-a):(s-b):(s-c)$

10. $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$; $\tan\frac{C-A}{2} = \frac{c-a}{c+a}.\cot\frac{B}{2}$; $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

11. D is a point on BC of triangle ABC dividing it in the ratio m:n. If $\angle ADC = \theta$, $\angle BAD = \alpha$, $\angle DAC = \beta$, then

i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

ii) $(m+n)\cot\theta = n\cot B - m\cot C$.



12. $\sin A + \sin B + \sin C = \frac{s}{R}$.

13. $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$

14. The distance of the circumcentre from BC is R cos A, from AB R cosC, from AC RcosB.

15. The distance of the orthocentre from BC is 2R cos BcosC, from AC is 2RcosAcosC, from AB is 2RcosAcosB

16. If H is the orthocenter of ΔABC, then HA = 2R cosA, HB = 2R cosB, HC = 2R cosC

17. $r = \frac{\Delta}{S} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

18. $r_1 = \dfrac{\Delta}{s-a} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = s\tan\dfrac{A}{2}$

19. $r_2 = \dfrac{\Delta}{s-b} = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2} = s\tan\dfrac{B}{2}$

20. $r_3 = \dfrac{\Delta}{s-c} = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2} = s\tan\dfrac{C}{2}$

21. The incentre of triangle ABC is the orthocentre of the excentral triangle.

22. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$

23. $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$

24. $r r_1 r_2 r_3 = \Delta^2$

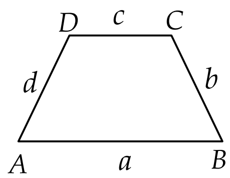
25. $\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} = a^2 + b^2 + c^2 + \Delta^2$

26. $r_1 - r = 4R\sin^2\dfrac{A}{2}$, $r_2 + r_3 = 4R\cos^2\dfrac{A}{2}$

27. $r_1 + r_2 + r_3 - r = 4R$

Area of a cyclic quadrilateral:

Area of cyclic quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$
Where $2s = a+b+c+d$
and $\cos B = \dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$



Regular Polygon

Radius of the inscribed circle of a regular polygon is $r = \dfrac{a}{2}\cot\left(\dfrac{\pi}{n}\right)$

Radius of the circumscribed circle of a regular polygon is $R = \dfrac{a}{2}\operatorname{cosec}\left(\dfrac{\pi}{n}\right)$

Where 'a' is the length of the side and 'n' is the number of sides of a polygon

and area of regular polygon is $\dfrac{1}{2}nR^2\sin\left(\dfrac{2\pi}{n}\right)$

Pedal triangle:

ABC is a triangle. D, E, F are feet of the perpendicular of altitudes of A, B, C on opposite sides then triangle DEF is called pedal triangle of ABC.

1. Circum radius of pedal triangle is $\frac{R}{2}$.

2. Angles of pedal triangle are $180° - 2A,\ 180° - 2B,\ 180° - 2C$

3. Sides of pedal triangle are $a\cos A, b\cos B$ and $cCosC$ or $R\sin 2A,\ R\sin 2B$ and $R\sin 2C$

4. In radius of the pedal triangle = $2R\cos A\cos B\cos C$

5. Area of the pedal triangle = $\frac{1}{2}R^2\sin 2A.\sin 2B.\sin 2C$

6. Ortho centre of ΔABC is incentre of pedal triangle

Ex-central triangle:

ABC be a traingle and I_1, I_2, I_3 are ex-centres then $\Delta I_1I_2I_3$ is called Ex-central triangle

1. Angles of excentral triangle are $90° - \frac{A}{2}, 90° - \frac{B}{2}, 90° - \frac{C}{2}$

2. Sides of excentral triangle are $4R\cos\frac{A}{2}, 4R\cos\frac{B}{2}, 4R\cos\frac{C}{2}$

3. Area of excentral triangle are $8R^2\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$

4. Circum radius of excentral triangle are $2R$

5. ABC is the pedal triangle of $I_1I_2I_3$.

$II_1 = 4R\sin\frac{A}{2}, II_2 = 4R\sin\frac{B}{2}, II_3 = 4R\sin\frac{C}{2}$

$AI = 4R\sin\frac{B}{2}.\sin\frac{C}{2}$, $BI = 4R\sin\frac{A}{2}.\sin\frac{C}{2}$, $CI = 4R\sin\frac{A}{2}.\sin\frac{B}{2}$

If O is the circumcentre and H is the orthocenter

$OI_1 = \sqrt{R^2 + 2Rr_1} = R\sqrt{1 + 8\sin\frac{A}{2}\cos\frac{B}{2}.\cos\frac{C}{2}}$

$OH = R\sqrt{1 - 8\cos A\cos B\cos C}$

$OI = \sqrt{R^2 - 2Rr}$

If I is the incentre of ΔABC then

$IA = r\cos ec\frac{A}{2}$, $IB = r\cos ec\frac{B}{2}$, $IC = r\cos ec\frac{C}{2}$

Length of the medians of ΔABC

$AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$, $BE = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$, $CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$